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FAST TRACK COMMUNICATION

On two (not so) new integrable partial difference equations

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Abstract
We examine two integrable discrete lattice equations obtained by Levi and Yamilov. We show that the first one is a form of the lattice KdV equation already obtained by Hirota and Tsujimoto, while the second one is a discrete form of mKdV. We present the Miura transformations between the various equations involved, including the more familiar potential form of the lattice mKdV.

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In a recent article [1], Levi and Yamilov derived two partial difference equations. Their construction was based on the combination of Miura transformations linking the Volterra differential-difference equation to the modified Volterra, for the first one, and to the doubly modified Volterra, for the second one. The authors of [1] explicitly showed the integrability of these systems by constructing their Lax pairs. Moreover, they claimed that these equations do not belong to the Adler–Bobenko–Suris classification given in [2] but that they are 3D consistent in a newer version [3] of this classification. In what follows, we analyse the two lattice equations of Levi and Yamilov and show that the first equation is a discrete form of KdV, obtained by Hirota and Tsujimoto in [4] where it was dubbed discrete Lotka–Volterra, and that the second equation is a discrete version of the modified KdV.

The first equation of Levi and Yamilov has the form

$$(u_{m+1,n} + 1)(u_{m,n} - 1) = (u_{m+1,n+1} - 1)(u_{m,n+1} + 1). \tag{1}$$

We start our analysis by examining its singularity pattern. Since (1) is an integrable equation, we expect its singularities to be confined [5]. It turns out that this is indeed the case. A straightforward calculation leads to the following pattern:

$$\begin{array}{ccccc} 1 & - & \infty & - & -1 \\ | & & | & & | \\ -1 & - & \infty & - & 1. \end{array}$$

This means that, if at some point (m, n) on the lattice $u_{m,n}$ happens to be equal to -1, from equation (1) it follows that $u_{m,n+1} = 1$, both $u_{m+1,n}$ and $u_{m+1,n+1}$ diverge, $u_{m+2,n} = 1, u_{m+2,n+1} = -1$ and all the subsequent u's are finite. Based on the singularity pattern, we can now proceed to the bilinearization of equation (1). As we have explained in [6] the existence of a single singularity pattern is expected to be associated with the existence of a single τ-function. With a little hindsight, we introduce the following ansatz:

$$u_{m,n} = -1 + \frac{F_{m-1,n-1}F_{m+1,n}}{F_{m,n-1}F_{m,n}} = 1 - \frac{F_{m+1,n-1}F_{m-1,n}}{F_{m,n-1}F_{m,n}}. \tag{2}$$

Substituting appropriate combinations of this ansatz into (1), the latter is identically verified. Moreover, from the second equality in (2) we obtain the following bilinear equation:

$$2F_{m,n-1}F_{m,n} - F_{m-1,n-1}F_{m+1,n} - F_{m+1,n-1}F_{m-1,n} = 0. \tag{3}$$

We should point out here that the coefficients of (3) do not have any deep meaning. Indeed, introducing a quadratic gauge of the form $F \to \mathrm{e}^{\alpha m^2+\beta mn+\gamma n^2}F$ we may assign to these coefficients any value we wish.

Equation (3) is nothing but the bilinear form of the discrete KdV equation [7]. In [8], we examined integrable lattice equations and derived the discrete form of KdV, starting from the Hirota–Miwa equation. The form obtained was

$$z_1 f_{m,n-1}f_{m,n} + z_2 f_{m-1,n-1}f_{m+1,n} + z_3 f_{m+1,n-1}f_{m-1,n} = 0, \tag{4}$$

i.e. precisely equation (3) up to a gauge which, as explained above, allows one to assign arbitrary values to the coefficients. We, therefore, claim that equation (1) of Levi and Yamilov is just a discrete analogue of KdV. Moreover, it is not a new form but one that has been known for quite some time. Indeed, in [4] Hirota and Tsujimoto have presented three integrable lattice equations related to KdV. The first was the 'standard' discrete KdV equation,

$$x_{m,n+1} - x_{m+1,n-1} = \frac{1}{x_{m+1,n}} - \frac{1}{x_{m,n}}. \tag{5}$$

The two remaining equations were presented as the 'discrete Lotka–Volterra of type I',

$$w_{m+1,n} - w_{m,n} = w_{m+1,n}w_{m+1,n-1} - w_{m,n}w_{m,n+1}, \tag{6}$$

and the 'discrete Lotka–Volterra of type II',

$$\frac{W_{m+1,n}}{(1+W_{m+1,n-1})(1+W_{m+1,n})} = \frac{W_{m,n}}{(1+W_{m,n})(1+W_{m,n+1})}. \tag{7}$$

The two Lotka–Volterra equations are related in an elementary way. In fact, it suffices to put

$$w_{m,n} = \frac{W_{m,n}}{1+W_{m,n}} \tag{8}$$

in the discrete Lotka–Volterra of type I in order to recover the equation of type II.

Focussing on equation (6), we rearrange its terms and obtain

$$w_{m+1,n}(1 - w_{m+1,n-1}) = w_{m,n}(1 - w_{m,n+1}). \tag{9}$$

We then perform a change of axes, introducing new indices, $N \equiv m$, $M = n + m$, after which equation (9) is rewritten as

$$w_{M+1,N+1}(1 - w_{M,N+1}) = w_{M,N}(1 - w_{M+1,N}). \tag{10}$$

Finally, we introduce $u = 1 - 2w$, whereupon (10) becomes identical to the equation of Levi and Yamilov (where the indices must be understood as capitalized to M, N). Thus equation (1) is just what Hirota and Tsujimoto call the discrete Lotka–Volterra equation although, as we have seen, based on the bilinear form the proper name should be that of the discrete KdV.

At this point, it is natural to ask the question whether this form of the discrete KdV is related to the standard one, i.e. equation (5). First we rotate the axes as before and obtain from discrete KdV the equation

$$x_{M+1,N} - x_{M,N+1} = \frac{1}{x_{M+1,N+1}} - \frac{1}{x_{M,N}}. \tag{11}$$

Using the relations given by Hirota and Tsujimoto, we introduce the Miura:

$$u_{M,N} = \frac{1 - x_{M,N}x_{M+1,N}}{1 + x_{M,N}x_{M+1,N}}. \tag{12}$$

When x satisfies the standard discrete KdV (11), u satisfies equation (1) with capitalized indices.

We now turn to the second equation of Levi and Yamilov which has the form

$$(1 + v_{m,n}v_{m+1,n})(\mu v_{m+1,n+1} + v_{m,n+1}/\mu) = (1 + v_{m,n+1}v_{m+1,n+1})(\mu v_{m,n} + v_{m+1,n}/\mu). \tag{13}$$

As in the case of the discrete KdV, we start with its singularity analysis, which yields the following patterns:

$$\begin{array}{ccc} \pm 1/\mu & - & \mp\mu \\ | & & | \\ \pm\mu & - & \mp 1/\mu. \end{array}$$

Since two different patterns are present we shall try to bilinearize (13) using two τ-functions. We introduce F and G—associated with the above singularity pattern with the upper and the lower sign, respectively—and require that the dependent variable $u_{m,n}$ take the value μ when either F vanishes at the point $(m-1, n-1)$ or G vanishes at (m, n). Furthermore, if F and G are interchanged, u takes the value $-\mu$. The appropriate ansatz now becomes

$$v_{m,n} = \mu\left(1 + \frac{F_{m-1,n-1}G_{m,n}}{\Delta}\right) = -\mu\left(1 + \frac{F_{m,n}G_{m-1,n-1}}{\Delta}\right), \tag{14}$$

and thus $\Delta = -(F_{m,n}G_{m-1,n-1} + F_{m-1,n-1}G_{m,n})/2$. We can rewrite the above ansatz as

$$v_{m,n} = \mu\left(\frac{F_{m,n}G_{m-1,n-1} - F_{m-1,n-1}G_{m,n}}{F_{m,n}G_{m-1,n-1} + F_{m-1,n-1}G_{m,n}}\right). \tag{15a}$$

Similarly, for the points where u takes the value $\pm 1/\mu$ we have also

$$v_{m,n} = \frac{1}{\mu}\left(\frac{F_{m,n-1}G_{m-1,n} - F_{m-1,n}G_{m,n-1}}{F_{m,n-1}G_{m-1,n} + F_{m-1,n}G_{m,n-1}}\right). \tag{15b}$$

The bilinear equations are obtained by equating the right hand sides of (15a) and (15b). Again, using the appropriate combination of (15a) and (15b), equation (13) is automatically satisfied. It is convenient to separate the equations as follows:

$$\mu^2(F_{m,n}G_{m-1,n-1} - F_{m-1,n-1}G_{m,n}) = F_{m,n-1}G_{m-1,n} - F_{m-1,n}G_{m,n-1}, \tag{16a}$$

$$F_{m,n}G_{m-1,n-1} + F_{m-1,n-1}G_{m,n} = F_{m,n-1}G_{m-1,n} + F_{m-1,n}G_{m,n-1}. \tag{16b}$$

Clearly some arbitrariness enters at this point but this can be remedied by a gauge transformation. Next we add and subtract (16a) and (16b), and find

$$F_{m-1,n-1}G_{m,n}(1-\mu^2) + F_{m,n}G_{m-1,n-1}(1+\mu^2) = 2F_{m,n-1}G_{m-1,n}, \tag{17a}$$

$$F_{m-1,n-1}G_{m,n}(1+\mu^2) + F_{m,n}G_{m-1,n-1}(1-\mu^2) = 2F_{m-1,n}G_{m,n-1}, \tag{17b}$$

or equivalently

$$2\mu^2 F_{m-1,n-1}G_{m,n} = F_{m,n-1}G_{m-1,n}(\mu^2-1) + F_{m-1,n}G_{m,n-1}(\mu^2+1), \tag{18a}$$

$$2\mu^2 F_{m,n}G_{m-1,n-1} = F_{m,n-1}G_{m-1,n}(\mu^2+1) + F_{m-1,n}G_{m,n-1}(\mu^2-1). \tag{18b}$$

We can now compare these bilinear equations to those obtained in [9] for the discrete modified KdV:

$$(\epsilon+\delta)G_{m-1,n}F_{m,n-1} + (\epsilon-\delta)G_{m,n-1}F_{m-1,n} - 2\epsilon G_{m,n}F_{m-1,n-1} = 0, \tag{19a}$$

$$(\epsilon+\delta)G_{m,n-1}F_{m-1,n} + (\epsilon-\delta)G_{m-1,n}F_{m,n-1} - 2\epsilon G_{m-1,n-1}F_{m,n} = 0. \tag{19b}$$

Taking $\epsilon = \mu^2, \delta = -1$, we remark that equations (19) coincide with those of (18). Thus, on the basis of the bilinear form we can conclude that equation (13) of Levi and Yamilov is a discrete analogue of the modified KdV equation.

A discrete form of mKdV is already known. Indeed, Capel *et al* [10] have presented a nonlinear form of what they call the *potential* discrete mKdV. It has the form

$$(\epsilon-\delta)(w_{m,n}w_{m-1,n} - w_{m,n-1}w_{m-1,n-1}) + (\epsilon+\delta)(w_{m,n}w_{m,n-1} - w_{m-1,n}w_{m-1,n-1}) = 0. \tag{20}$$

Its bilinear form (19) was obtained in [9] through the ansatz $w = F/G$. However, reordering the terms in (20) we find

$$\mu\frac{w_{m,n} - w_{m-1,n-1}}{w_{m,n} + w_{m-1,n-1}} = \frac{1}{\mu}\frac{w_{m,n-1} - w_{m-1,n}}{w_{m,n-1} + w_{m-1,n}}, \tag{21}$$

which is clearly not the equation of Levi and Yamilov. The fact that the bilinear form of the latter is the same as that of (20) is not a problem: the bilinear forms do not distinguish between the standard and the potential form of some equation; only the ansatz for the variable of the nonlinear equation in terms of the τ-functions is different. Comparing now (21) to the right-hand sides of (15*a*) and (15*b*) (using the bilinear ansatz $w = F/G$) it is clear that equation (21) merely expresses the fact that the left-hand sides of (15*a*) and (15*b*) are identical. Moreover, using (15) we can give the Miura transformation between the potential mKdV and the equation of Levi and Yamilov. We find

$$v_{m,n} = \mu\frac{w_{m,n} - w_{m-1,n-1}}{w_{m,n} + w_{m-1,n-1}} = \frac{1}{\mu}\frac{w_{m,n-1} - w_{m-1,n}}{w_{m,n-1} + w_{m-1,n}}. \tag{22}$$

At this point a short digression is in order. The potential mKdV of Capel and collaborators is not the only one appearing in the literature under this name. Indeed, in [11] Hirota has derived another form of the discrete potential mKdV, based on his bilinear formalism. The equation of Hirota has the form

$$\tan\left(\phi_j^{k+\frac{1}{2}} - \phi_j^{k-\frac{1}{2}}\right) = \epsilon \tan\left(\phi_{j-\frac{1}{2}}^k - \phi_{j+\frac{1}{2}}^k\right). \tag{23}$$

While (23) is at first sight different from (20), the two equations are in fact identical. First, we expand both sides of equation (23) as

$$\frac{\tan\phi_j^{k+\frac{1}{2}} - \tan\phi_j^{k-\frac{1}{2}}}{1 + \tan\phi_j^{k+\frac{1}{2}}\tan\phi_j^{k-\frac{1}{2}}} = \epsilon\frac{\tan\phi_{j-\frac{1}{2}}^k - \tan\phi_{j+\frac{1}{2}}^k}{1 + \tan\phi_{j-\frac{1}{2}}^k\tan\phi_{j+\frac{1}{2}}^k}. \tag{24}$$

Next we introduce the variable $w = \frac{1+\mathrm{i}\tan\phi}{1-\mathrm{i}\tan\phi} \equiv \mathrm{e}^{2\mathrm{i}\phi}$ and transform (24) into

$$\frac{w_j^{k-\frac{1}{2}} - w_j^{k+\frac{1}{2}}}{w_j^{k-\frac{1}{2}} + w_j^{k+\frac{1}{2}}} = \epsilon\frac{w_{j-\frac{1}{2}}^k - w_{j+\frac{1}{2}}^k}{w_{j-\frac{1}{2}}^k + w_{j+\frac{1}{2}}^k}. \tag{25}$$

It suffices now to take $\epsilon = \mu^2$ and change the axes so as to have $m = j + k + 1/2$, $n = j - k + 1/2$, in order to obtain equation (21).

We complete our analysis by showing how the two equations of Levi and Yamilov are related, which is something we believe to be of practical interest. Since these equations are discrete forms of KdV and mKdV, we expect a Miura transformation to exist between them. In order to find this transformation we again rely on the bilinear formalism. First we remark that, using equation (16), the bilinear form of a certain combination of $v_{m,n}$ and $v_{m+1,n}$ can be expressed in terms of the τ-function F only. We have

$$\frac{(\mu - v_{m,n})(\mu + v_{m+1,n})}{(1 - \mu v_{m,n})(1 + \mu v_{m+1,n})} = \mu^2 \frac{F_{m-1,n-1}F_{m+1,n}}{F_{m-1,n}F_{m+1,n-1}}. \tag{26}$$

Next, using the ansatz (2) we can express the rhs of (26) in terms of u. However, if one eliminates G from (16*a*), (16*b*) and their appropriate up- and down-shifts, one finds an equation of form (4) with coefficients different from those in (3). This means that the τ-function, F, here is not exactly that of equations (2) and (3) but, as pointed out when equation (3) was derived, this just means that the particular gauge in which (16*a*) and (16*b*) are written differs from that of (2) and (3). Correcting for this difference in gauge, we finally obtain

$$\frac{u_{m,n} + 1}{u_{m,n} - 1} = \frac{(\mu - v_{m,n})(\mu + v_{m+1,n})}{(1 - \mu v_{m,n})(1 + \mu v_{m+1,n})}, \tag{27}$$

and, solving for u,

$$u_{m,n} = \frac{(1 + \mu^2)(v_{m,n}v_{m+1,n} - 1) + 2\mu(v_{m,n} - v_{m+1,n})}{(1 - \mu^2)(v_{m,n}v_{m+1,n} + 1)}, \tag{28}$$

which is, as can easily be verified, the Miura transformation relating the two equations of Levi and Yamilov.

Before concluding this paper it seems necessary to justify the names attributed to the equations, in particular, since we claim that equation (13) is a discrete form of mKdV, and not of its potential variant. This can be done by proceeding to the continuous limit. We start with equation (1) which is a discrete form of KdV. First we introduce the ansatz $u = \kappa + \varepsilon^2 U$. Next we choose a shift in the direction m to correspond to a pure space evolution, i.e. $u_{m+1,n} = \kappa + \varepsilon^2(U + \varepsilon U_x + \varepsilon^2 U_{xx}/2 + \varepsilon^3 U_{xxx}/6 + \cdots)$. Similarly, a shift in the direction n corresponds to an evolution in both x and t, the corresponding velocity being c/ε^2. We have thus $u_{m,n+1} = \kappa + \varepsilon^2(U + c\varepsilon U_x + c^2\varepsilon^2 U_{xx}/2 + c^3\varepsilon^3 U_{xxx}/6 + \varepsilon^3 U_t + \cdots)$. In order to go to the continuous limit we take $\varepsilon \to 0$. We find first that c and κ are related through $c\kappa + 1 = 0$. Since κ is a free parameter, we fix it to $\kappa = 1/3$ (which means that $c = -3$) in order to simplify the form of the final equation. We thus obtain

$$U_t - 9UU_x + 2U_{xxx} = 0, \tag{29}$$

which is obviously the KdV equation. In the case of equation (13) the ansatz is $v = \varepsilon V$. We choose the evolutions in the m and n directions just as we did previously, where the speed c is now related to μ. We find $c = (1 - \mu^2)/(1 + \mu^2)$. Again we simplify the final equation by assigning a specific value to μ. Taking $\mu = \mathrm{i}\sqrt{2}$ (which means that $c = -3$) we find

$$V_t + 6V^2 V_x + 2V_{xxx} = 0, \tag{30}$$

which is clearly the modified KdV equation in its standard form. Hence the naming of the two equations is justified on the basis of the continuous limit (as is customary in the discrete domain). We should also point out that the continuous form of the Miura transformation (28), relating the two equations of Levi and Yamilov, is just the usual Miura transformation linking KdV and mKdV. We indeed find

$$U = 2\frac{1+\mu^2}{1-\mu^2}V^2 - \frac{2\mu}{1-\mu^2}V_x, \tag{31}$$

provided c, μ and κ are related through the expressions found above.

Thus the equations obtained by Levi and Yamilov are discrete forms of KdV and mKdV, respectively. While the first one has been known for quite some time, the second one is apparently new. What is interesting is that the equations analysed here are connected, through Miura transformations, not only to each other but also to other integrable discrete equations like the 'standard' lattice KdV or the potential lattice mKdV equations.

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